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SOME NEW FIXED POINT RESULTS IN b-METRIC SPACES

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Abstract
In this study, we establish some fixed point theorems for novel contractive-type conditions for mappings in b-metric spaces. Some well-known fixed point-related results in the literature are improved, generalized and extended by our results.
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1. INTRODUCTION

Bakhtin [2] introduced the notion of b-metric space in 1989 and Czerwik [8] provided a formal definition in 1993. b-metric spaces are a generalization of metric spaces where the notion of distance satisfies certain properties akin to those of metrics, but not necessarily all. They are useful in extending various results from metric spaces to a broader setting.

One notable difference between b-metric spaces and traditional metric spaces is that the b-metric function may not necessarily be continuous within the topology it generates. This property distinguishes b-metric spaces from metric spaces and can have implications for the applicability and generalization of certain theorems. For instance, Example 2.6 of reference [16] likely provides a specific instance where the b-metric d is not continuous in the associated topology, illustrating a characteristic feature of b-metric spaces.

Researchers have explored various aspects of b-metric spaces and have contributed to the generalization of fixed point theorems in this context. References [3, 6, 9, 11, 12, 13, 14, 17], along with their respective citations, likely discuss these developments and related works in the field.

The unique fixed point theorem for weakly-contractive mappings was recently established by Chaudhary et al. [7]. Fixed point theory for multivalued generalized contraction on an asset with two b-metric spaces was demonstrated by Boriceanu [5]. A fixed point theorem for set valued quasi-contractions in b-metric space was established by Aydi et al. [1]. Berinde [4] proved Iterative Approximation of fixed points. In this attempt we establish fixed point theorems using new contractive type conditions for mappings defined on b-metric space. Our results are extensions of the findings in [15].

2. PRELIMINARIES

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Definition 2.1[2] : Let X be a non-empty set and let $s \geq 1$ be a given real number. A function $d: X \times X \to R_+$, is called a b-metric provided that, for all $p, q, r \in X$,

1) $d(p,q) = 0$ iff $p = q$,
2) $d(p,q) = d(q,p)$,
3) $\mathrm{d}(\mathrm{p},\mathrm{r}) \leq s[d(p,q) + d(q,r)]$.

A pair (X,d) is called a b-metric space. It is clear that definition of b-metric space is extension of usual metric space.

Example 2.1[5] : The space $l_x(0 < x < 1)$,

$l_x = \{(p_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |p_n|^x < \infty\}$,

together with the function $d: l_x \times l_x \to \mathbb{R}$

$d(p,q) = (\sum_{n=1}^{\infty} |p_n - q_n|^x)^{\frac{1}{x}}$,

Where $p = p_{n,}\ q = q_n \in l_x$ is a b-metric space. By an elementary calculation we obtain that

$\mathrm{d}(\mathrm{p},\mathrm{r}) \leq \frac{1}{2^x}[d(p,q) + d(q,r)]$.

Example 2.2 [5] : The $L_x(0 < x < 1)$ of all real functions $p(t), t \in [0,1]$ such that $\int_0^1 |p(t)|^x dt < \infty$, is b-metric space if we take

$d(p,q) = \left[\int_0^1 |p(t) - q(t)|^x dt\right]^{\frac{1}{x}}$,

for each $p, q \in L_x$.

Definition 2.2[5] : Let (X,d) be a metric space. Then a sequence $\{p_n\}$ in X is called a Cauchy sequence if and only if for all $\varepsilon > 0$ there exist $n(\varepsilon) \in N$ such that for each $n, m \geq n(\varepsilon)$ we have $d(p_n, p_m) < \varepsilon$.

Definition 2.3[5] : Let (X,d) be a metric space. Then a sequence $\{p_n\}$ in X is called a convergent sequence if and only if $p \in X$, then there exist $n(\varepsilon) \in N$ such that for all $n \geq n(\varepsilon)$ we have $d(p_n, p) < \varepsilon$. In this case, we write $\lim_{n\to\infty} p_n = p$.

Definition 2.4[5] : The b-metric space is complete if every Cauchy sequence convergent.

Definition 2.5[4] : Let E be a non-empty set and $T : E \to E$ a self map. We say that $p \in E$ is a fixed point of T if $T(p) = p$ and denote by FT or $Fip(T)$ the set of all fixed point of T. Let E be any set and $T : E \to E$ a self map.

For any given $p \in E$ we define $T^n(p)$ inductively by $T^0(p) = p$ and $T^{n+1}(p) = T(T^n(p))$, we recall $T^n(p)$, the n^{th} iterative of p under T. For any $p_0 \in X$, the sequence $\{p_n\}_{n\geq 0} \subset X$ given by

$p_n = Tp_{n-1} = T^n p_0, n = 1,2, ...$

is called the sequence of successive approximations with the initial value p_0. It is also known as the Picard iteration starting at p_0.

Definition 2.6[4] : Let (X,d) be metric space. A mapping $T : X \to X$ is called weak contraction if there exists a constant $\delta \in (0,1)$ and some $L \geq 0$ such that

$$d(Tp,Tq) \leq \delta d(p,q) + Ld(q,Tp) \tag{2.1}$$

Remark 2.7[4] : Due to symmetry of the distance, the weak contractive condition (2.1) imply implicity includes the following dual one

$$d(Tp,Tq) \leq \delta d(p,q) + Ld(p,Tq) \quad (2.2)$$

$\forall\ p,q \in X$.

In order to check the weak contractiveness of T; it is necessary to check both (2.1) and (2.2). It is clear that any contraction mapping is also weak contraction mapping in a metric space.

3. MAIN RESULTS

In this section, we give some fixed point theorems in b-metric space.

Theorem 3.1: Let (X,d) be a complete b-metric space with constant $s \geq 1$ and define the sequence $\{p_n\}_{n=1}^{\infty} \subset X$ by the recursion

$p_n = Tp_{n-1} = T^n p_0$.

Let $T: X \to X$ be a mapping such that

$$d(Tp,Tq) \leq \alpha_1 d(p,q) + \alpha_2 d(q,Tp) + \alpha_3 d(p,Tq) + \alpha_4[d(p,Tp) + d(q,Tq)], \quad (3.1)$$

where $\alpha_1 + \alpha_2 + 2s\alpha_3 + 2\alpha_4 \leq 1$

$\forall\ p,q \in X$ then there exists $p^* \in X$ such that $p_n \to p^*$ and p^* is a unique fixed point.

Proof: Let $p_0 \in X$ and $\{p_n\}_{n=1}^{\infty} \subset X$ be a sequence in X defined as

$$p_n = Tp_{n-1} = T^n p_0, n = 1,2,3, \dots \quad (3.2)$$

By (3.1) and (3.2) we obtain that

$$\begin{aligned} d(p_n,p_{n+1}) &= d(Tp_{n-1},Tp_n) \\ &\leq \alpha_1 d(p_{n-1},p_n) + \alpha_2 d(p_n,Tp_{n-1}) + \alpha_3 d(p_{n-1},Tp_n) + \alpha_4[d(p_{n-1},Tp_{n-1}) + d(p_n,Tp_n)] \\ &\leq \alpha_1 d(p_{n-1},p_n) + \alpha_2 d(p_n,p_n) + \alpha_3 d(p_{n-1},p_{n+1}) + \alpha_4[d(p_{n-1},p_n) + d(p_n,p_{n+1})] \\ &\leq \alpha_1 d(p_{n-1},p_n) + s\alpha_3[d(p_{n-1},p_n) + d(p_n,p_{n+1})] + \alpha_4 d(p_{n-1},p_n) + \alpha_4 d(p_n,p_{n+1}) \\ &\leq \alpha_1 d(p_{n-1},p_n) + s\alpha_3 d(p_{n-1},p_n) + s\alpha_3 d(p_n,p_{n+1}) + \alpha_4 d(p_{n-1},p_n) + \alpha_4 d(p_n,p_{n+1}) \\ &\leq (\alpha_1 + s\alpha_3 + \alpha_4) d(p_{n-1},p_n) + (s\alpha_3 + \alpha_4) d(p_n,p_{n+1}) \end{aligned}$$

$$\begin{aligned} \Rightarrow (1 - s\alpha_3 - \alpha_4) d(p_n,p_{n+1}) &\leq (\alpha_1 + s\alpha_3 + \alpha_4) d(p_{n-1},p_n) \\ &\leq kd(p_{n-1},p_n) \end{aligned}$$

where $k = \frac{\alpha_1 + s\alpha_3 + \alpha_4}{1 - s\alpha_3 - \alpha_4} \leq 1$

As $\alpha_1 + \alpha_2 + 2s\alpha_3 + 2\alpha_4 \leq 1$,

$$\alpha_1 + s\alpha_3 + \alpha_4 \leq 1 - s\alpha_3 - \alpha_4$$

$$\frac{\alpha_1 + s\alpha_3 + \alpha_4}{1 - s\alpha_3 - \alpha_4} \leq 1$$

$d(p_n,p_{n+1}) \leq kd(p_{n-1},p_n)$

$\leq k^2 d(p_{n-2}, p_{n-1})$

Continuing this process, we get

$\leq k^n d(p_0, p_1)$.

Now, we show that $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Let $m, n > 0$ with $m > n$

$$d(p_n, p_m) \leq sd(p_n, p_{n+1}) + s^2 d(p_{n+1}, p_{n+2}) + s^3 d(p_{n+2}, p_{n+3}) + - - -$$
$$\leq sk^n d(p_1, p_0) + s^2 k^{n+1} d(p_1, p_0) + - - - + s^m k^{n+m-1} d(p_1, p_0)$$
$$\leq sk^n d(p_1, p_0)[1 + (sk) + (sk)^2 + - - - + (sk)^{m-1}]$$
$$\leq sk^n d(p_1, p_0)\left[\frac{1-(sk)^{n-(m-1)}}{1-sk}\right].$$

When we take $m, n \to \infty$

$\lim_{n\to\infty} d(p_n, p_m) = 0$.

Hence $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence, $\{p_n\}$ converges to $p^* \in X$.

Now we show that p^* is the unique fixed point of T.

$$d(p^*, Tp^*) \leq s[d(p^*, p_{n+1}) + d(p_{n+1}, Tp^*)]$$
$$\leq sd(p^*, p_{n+1}) + sd(Tp_n, Tp^*)$$
$$\leq sd(p^*, p_{n+1}) + s\alpha_1 d(p_n, p^*) + s\alpha_2 d(p^*, Tp_n)$$
$$+ s\alpha_3 d(p_n, Tp^*) + s\alpha_4[d(p_n, Tp_n) + d(p^*, Tp^*)]$$
$$\Longrightarrow d(p^*, Tp^*) \leq sd(p^*, p_{n+1}) + s\alpha_1 d(p_n, p^*) + s\alpha_2 d(p^*, p_{n+1})$$
$$+ s\alpha_3 d(p_n, Tp^*) + s\alpha_4 d(p_n, p_{n+1}) + s\alpha_4(p^*, Tp^*)]$$
$$\leq sd(p^*, p_{n+1}) + s\alpha_1 d(p_n, p^*) + s\alpha_2 d(p^*, p_{n+1}) + s^2\alpha_3 d(p_n, p^*)$$
$$+ s^2\alpha_3 d(p^*, Tp^*) + s^2\alpha_4 d(p_n, p^*)$$
$$+ s^2\alpha_4 d(p^*, p_{n+1}) + s\alpha_4 d(p^*, Tp^*)$$
$$\Longrightarrow (1 - s^2\alpha_3 - s\alpha_4) d(p^*, Tp^*) \leq sd(p^*, p_{n+1}) + s\alpha_1 d(p_n, p^*) + s\alpha_2 d(p^*, p_{n+1})$$
$$+ s^2\alpha_3 d(p_n, p^*) + s^2\alpha_4 d(p_n, p^*)$$
$$+ s^2\alpha_4 d(p^*, p_{n+1}).$$
$$\Rightarrow (1 - s^2\alpha_3 - s\alpha_4) d(p^*, Tp^*) \leq (s + s\alpha_2 + s^2\alpha_4) d(p_{n+1}, p^*)$$
$$+ (s\alpha_1 + s^2\alpha_3 + s^2\alpha_4) d(p_n, p^*)$$
$$\Rightarrow d(p^*, Tp^*) \leq \frac{(s + s\alpha_2 + s^2\alpha_4)}{1 - s^2\alpha_3 - s\alpha_4} d(p_{n+1}, p^*) + \frac{(s\alpha_1 + s^2\alpha_3 + s^2\alpha_4)}{1 - s^2\alpha_3 - s\alpha_4} d(p_n, p^*)$$

$d(p^*, Tp^*) \leq 0$ as $n \to \infty$. Now we show that p^* is the fixed point of T. Assume that p' is another fixed point of T, then we have $Tp' = p'$ and

$$d(p^*, p') = d(Tp^*, Tp')$$
$$\leq \alpha_1 d(p^*, p') + \alpha_2 d(p', Tp^*) + \alpha_3 d(p^*, Tp') + \alpha_4[d(p^*, Tp^*) + d(p', Tp')]$$
$$\leq \alpha_1 d(p^*, p') + \alpha_2 d(p', p^*) + \alpha_3 d(p^*, p') + \alpha_4[d(p^*, p^*) + d(p', p')]$$
$$\leq \alpha_1 d(p^*, p') + \alpha_2 d(p', p^*) + \alpha_3 d(p^*, p') = (\alpha_1 + \alpha_2 + \alpha_3) d(p^*, p')$$

which implies that $p^* = p'$.

Theorem 3.2: Let (X,d) be a complete b-metric space with constant $s \geq 1$. Let $T: X \to X$ be a mapping for which there exist $\alpha \in [0,\frac{1}{2})$ such that

$$d(Tp,Tq) \leq \alpha[d(p,q) + d(q,Tq)] \tag{3.3}$$

$\forall\, p,q \in X$ then there exists $p^* \in X$ such that $p_n \to p^*$ and p^* is a unique fixed point of T.

Proof: Let $p_0 \in X$ and $\{p_n\}_{n=1}^{\infty} \subset X$ be a sequence in X defined as

$$p_n = Tp_{n-1} = T^n p_0,\ \text{n} = 1,2,3,\ \ldots \tag{3.4}$$

By using (3.3) we obtain that

$$\begin{aligned} d(p_n,p_{n+1}) &= d(Tp_{n-1},Tp_n) \\ &\leq \alpha[d(p_{n-1},p_n) + d(p_n,Tp_n)] \\ &\leq \alpha[d(p_{n-1},p_n) + d(p_n,p_{n+1})] \\ &\leq \alpha d(p_{n-1},p_n) + \alpha d(p_n,p_{n+1}) \end{aligned}$$

$$\Rightarrow (1-\alpha)d(p_n,p_{n+1}) \leq \alpha d(p_{n-1},p_n)$$

$$\begin{aligned} d(p_n,p_{n+1}) &\leq \frac{\alpha}{1-\alpha} d(p_{n-1},p_n) \\ &\leq k d(p_{n-1},p_n) \end{aligned}$$

Where $k = \frac{\alpha}{1-\alpha} \leq 1$

As $2\alpha \leq 1$

$$\Rightarrow \alpha \leq 1-\alpha$$

$$\frac{\alpha}{1-\alpha} \leq 1$$

$$d(p_n,p_{n+1}) \leq k^n d(p_1,p_0)$$

Thus T is a contraction mapping.

Now, we show that $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Let $m,n > 0$ with $m > n$

$$\begin{aligned} d(p_n,p_m) &\leq s[d(p_n,p_{n+1}) + d(p_{n+1},p_m)] \\ &\leq sd(p_n,p_{n+1}) + s^2 d(p_{n+1},p_{n+2}) + s^3 d(p_{n+2},p_{n+3}) \\ &\quad + - - - + s^m d(p_{n+m-1},p_m) \end{aligned}$$

$$d(p_n,p_m) \leq sk^n d(p_1,p_0) + s^2 k^{n+1} d(p_1,p_0) + - - - + s^m k^{n+m-1} d(p_1,p_0)$$

$$\leq (sk^n) d(p_1,p_0)[1 + (sk) + (sk)^2 + (sk)^{m-1}]$$

$$\leq (sk^n) d(p_1,p_0)[\frac{1-(sk)^{n-(m-1)}}{1-sk}]$$

when we take $m,n \to \infty$

$\lim_{n\to\infty} d(p_n,p_m) = 0$.

Hence $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence, $\{p_n\}$ converges to $p^* \in X$.

Now we show that p^* is the unique fixed point of T.

$d(p^*,Tp^*) \leq s[d(p^*,p_n) + d(p_n,Tp^*)]$

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$$d(p^*,Tp^*) \leq s[d(p^*,p_n) + d(Tp_{n-1},Tp^*)]$$
$$\leq sd(p^*,p_n) + sd(Tp_{n-1},Tp^*)$$
$$\leq sd(p^*,p_n) + s\alpha[d(p_{n-1},p^*) + d(p^*,Tp^*)]$$
$$\leq sd(p_n,p^*) + s\alpha d(p_{n-1},p^*) + s\alpha d(p^*,Tp^*)$$
$$\Rightarrow (1-s\alpha)\, d(p^*,Tp^*) \leq sd(p_n,p^*) + s\alpha d(p_{n-1},p^*)$$
$$\lim_{n\to\infty} d(p^*,Tp^*) = 0$$
$$i.e\; Tp^* = p^*$$

Now we show that p^* is the fixed point of T . Assume that p' is another fixed point of T, then we have $Tp' = p'$ and

$$d(p^*,p') = d(Tp^*,Tp')$$
$$\leq \alpha[d(p^*,p') + d(p',Tp')]$$
$$d(p^*,p') \leq \alpha d(p^*,p')$$
$$(1-\alpha)d(p^*,p') \leq 0$$

which implies that $p^* = p'$.
This completes the proof.

Theorem 3.3: Let (X,d) be a complete b-metric space with constant $s \geq 1$ and define the sequence $\{p_n\}_{n=1}^{\infty} \subset X$ by the recursion
$p_n = Tp_{n-1} = T^n p_0$.
Let $T : X \to X$ be a mapping such that

$$d(Tp,Tq) \leq \alpha_1 d(p,q) + \alpha_2 d(p,Tq) + \alpha_3 d(q,Tq) + \alpha_4[\, d(p,Tp) + d(q,Tq)], \tag{3.5}$$

where $\alpha_1 + 2s\alpha_2 + \alpha_3 + 2\alpha_4 \leq 1$ and $\forall\, p,q \in X$ then there exists $p^* \in X$ such that $p_n \to p^*$ and p^* is a unique fixed point.
Proof: Let $p_0 \in X$ and $\{p_n\}_{n=1}^{\infty} \subset X$ be a sequence in X defined as

$$p_n = Tp_{n-1} = T^n p_0,\ \text{n} = 1,2,3,\ldots \tag{3.6}$$

By (3.5) and (3.6) we obtain that

$$d(p_n,p_{n+1}) = d(Tp_{n-1},Tp_n)$$
$$\leq \alpha_1 d(p_{n-1},p_n) + \alpha_2 d(\,p_{n-1},Tp_n)$$
$$+ \alpha_3 d(p_n,Tp_n\,)$$
$$+ \alpha_4[d(p_{n-1}\,,Tp_{n-1})$$
$$+ d(p_n,Tp_n)]$$
$$= \alpha_1 d\,(p_{n-1},p_n) + s\alpha_2[d(\,p_{n-1},p_n\,) + d(p_n,p_{n+1})] + \alpha_3 d(p_n,p_{n+1})$$
$$+ \alpha_4[d(p_{n-1},p_n) + d(p_n,p_{n+1})]$$
$$\leq \alpha_1 d(p_{n-1},p_n) + s\alpha_2 d(\,p_{n-1},p_n) + s\alpha_2 d(p_n,p_{n+1}) + \alpha_3 d(p_n,p_{n+1})$$
$$+ \alpha_4[d(p_{n-1},p_n) + d(p_n,p_{n+1}\,)]$$
$$= (\alpha_1 + s\alpha_2 + \alpha_4)d(p_{n-1},p_n) + (s\alpha_2 + \alpha_3 + \alpha_4)d(p_n,p_{n+1})$$

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$$\Rightarrow (1 - s\alpha_2 - \alpha_3 - \alpha_4)d(p_n, p_{n+1}) \leq (\alpha_1 + s\alpha_2 + \alpha_4)d(p_{n-1}, p_n)$$

$\leq kd(p_{n-1}, p_n)$

where $k = \frac{\alpha_1 + s\alpha_2 + \alpha_4}{1 - s\alpha_2 - \alpha_3 - \alpha_4} \leq 1$

As $\alpha_1 + 2s\alpha_2 + \alpha_3 + 2\alpha_4 \leq 1$

$$\alpha_1 + s\alpha_2 + \alpha_4 \leq 1 - s\alpha_2 - \alpha_3 - \alpha_4$$

$$\frac{\alpha_1 + s\alpha_2 + \alpha_4}{1 - s\alpha_2 - \alpha_3 - \alpha_4} \leq 1$$

$d(p_n, p_{n+1}) \leq kd(p_{n-1}, p_n)$

$\leq k^2 d(p_{n-2}, p_{n-1})$

Continuing this process, we get

$d(p_n, p_{n+1}) \leq k^n d(p_0, p_1)$.

Now, we show that $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Let $m, n > 0$ with $m > n$,

$$d(p_n, p_{n+1}) \leq sd(p_n, p_{n+1}) + s^2 d(p_{n+1}, p_{n+2}) + s^3 d(p_{n+2}, p_{n+3}) + ---$$

$$\leq sk^n d(p_1, p_0) + s^2 k^{n+1} d(p_1, p_0) + --- + s^m k^{n+m-1} d(p_1, p_0)$$

$$\leq sk^n d(p_1, p_0)[1 + (sk) + (sk)^2 + --- + (sk)^{m-1}]$$

$$\leq sk^n d(p_1, p_0)\left[\frac{1 - (sk)^{n-(m-1)}}{1 - sk}\right].$$

When we take $m, n \to \infty$

$\lim_{n\to\infty} d(p_n, p_m) = 0$.

Hence $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence, $\{p_n\}$ converges to $p^* \in X$.

Now we show that p^* is the unique fixed point of T.

$$d(p^*, Tp^*) \leq s[d(p^*, p_{n+1}) + d(p_{n+1}, Tp^*)]$$

$$d(p^*, Tp^*) \leq sd(p^*, p_{n+1}) + sd(p_{n+1}, Tp^*)$$

$$\leq sd(p^*, p_{n+1}) + sd(Tp_n, Tp^*)$$

$$\leq sd(p^*, p_{n+1}) + s\alpha_1 d(p_n, p^*) + s\alpha_2 d(p_n, Tp^*) + s\alpha_3 d(p^*, Tp^*) + s\alpha_4[d(p_n, Tp_n) + d(p^*, Tp^*)]$$

$$\Longrightarrow d(p^*, Tp^*) \leq sd(p^*, p_{n+1}) + s\alpha_1 d(p_n, p^*) + s\alpha_2 d(p_n, Tp^*) + s\alpha_3 d(p^*, Tp^*) + s\alpha_4 d(p_n, p_{n+1}) + s\alpha_4 d(p^*, Tp^*)$$

$$\leq sd(p^*, p_{n+1}) + s\alpha_1 d(p_n, p^*) + s^2\alpha_2 d(p_n, p^*) + s^2\alpha_2 d(p^*, Tp^*) + s\alpha_3 d(p^*, Tp^*) + s^2\alpha_4 d(p_n, p^*) + s^2\alpha_4 d(p^*, p_{n+1}) + s\alpha_4 d(p^*, Tp^*)]$$

$$(1 - s^2\alpha_2 - s\alpha_3 - s\alpha_4)d(p^*, Tp^*) \leq sd(p^*, p_{n+1}) + s\alpha_1 d(p_n, p^*) + s^2\alpha_2 d(p^*, p_n) + s^2\alpha_4 d(p_n, p^*) + s^2\alpha_4 d(p^*, p_{n+1})$$

$$\Rightarrow (1 - s^2\alpha_2 - s\alpha_3 - s\alpha_4)d(p^*, Tp^*) \leq (s + s^2\alpha_4)d(p_{n+1}, p^*) + (s\alpha_1 + s^2\alpha_2 + s^2\alpha_4)d(p_n, p^*)$$

$$\Rightarrow d(p^*, Tp^*) \leq \frac{s + s^2\alpha_4}{1 - s^2\alpha_2 - s\alpha_3 - s\alpha_4} d(p_{n+1}, p^*) + \frac{s\alpha_1 + s^2\alpha_2 + s^2\alpha_4}{1 - s^2\alpha_2 - s\alpha_3 - s\alpha_4} d(p_n, p^*)$$

$d(p^*, Tp^*) \le 0$ as $n \to \infty$. Now we show that p^* is the fixed point of T. Assume that p' is another fixed point of T, then we have $Tp' = p'$ and

$$d(p^*, p') = d(Tp^*, Tp')$$
$$\le \alpha_1 d(p^*, p') + \alpha_2 d(p^*, Tp') + \alpha_3 d(p', Tp') + \alpha_4[d(p^*, Tp^*) + d(p', Tp')]$$
$$\le \alpha_1 d(p^*, p') + \alpha_2 d(p^*, p') + \alpha_3 d(p', p') + \alpha_4[d(p^*, p^*) + d(p', p')]$$
$$\le \alpha_1 d(p^*, p') + \alpha_2 d(p^*, p') = (\alpha_1 + \alpha_2) d(p^*, p')$$

which implies that $p^* = p'$.

Corollary 3.1: Let (X, d) be a complete b-metric space with constant $s \ge 1$ and define the sequence $\{p_n\}_{n=1}^{\infty} \subset X$ by the recursion
$p_n = Tp_{n-1} = T^n p_0$.
Let $T: X \to X$ be a mapping such that

$$d(Tp, Tq) \le \alpha_1 d(p, q) + \alpha_2 d(q, Tp) + \alpha_3 d(p, Tq) + \alpha_4 d(p, Tp) + \alpha_5 d(q, Tq) \tag{3.7}$$

where $\alpha_1 + \alpha_2 + 2s\alpha_3 + \alpha_4 + \alpha_5 \le 1$ and $\forall\, p, q \in X$, then there exists $p^* \in X$ such that $p_n \to p^*$ and p^* is a unique fixed point.

Proof: It is the modify from of Theorem 3.1 by introducing α_5.

Corollary 3.2: Let (X, d) be a complete b-metric space with constant $s \ge 1$. Let $T: X \to X$ be a mapping for which there exist $\alpha_1, \alpha_4 \in \left[0, \frac{1}{3}\right)$ such that

$$d(Tp, Tq) \le \alpha_1 d(p, q) + \alpha_4[d(p, Tp) + d(q, Tq)] \tag{3.8}$$

$\forall\, p, q \in X$, then there exists $p^* \in X$ such that $p_n \to p^*$ and p^* is a unique fixed point of T.

Proof: By putting $\alpha_2 = \alpha_3 = 0$ in Theorem 3.1, we get the result of Corollary 3.2, which is the main result (Theorem 2) of Dubey et al. [10].

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